

RECTANGLE GELL-MANN MATRICES

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Abstract

We call rectangles Gell-Mann matrices rectangle matrices which make generalization of the expression of a tensor commutation matrix $n \otimes n$ in terms of tensor products of square Gell-Mann matrices.

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1 Introduction

For any $n \in \mathbb{N}$, $n \geq 2$, a tensor commutation matrix $n \otimes n$, can be expressed in terms of $n \times n$ -Gell-Mann matrices $\Lambda_a^{(n)}$ for the following way [1]

$$U_{n \otimes n} = \frac{1}{n} I_n \otimes I_n + \frac{1}{2} \sum_{a=1}^{n^2-1} \Lambda_a^{(n)} \otimes \Lambda_a^{(n)} \quad (1.1)$$

where I_n is the $n \times n$ unit matrix. Particularly, for $n = 2$, $U_{2 \otimes 2}$ can be expressed in terms of the Pauli matrices, namely

$$U_{2 \otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} I_2 \otimes I_2 + \frac{1}{2} \sum_{i=1}^3 \sigma_i \otimes \sigma_i$$

which is frequently found in quantum information theory [2], [3], [4].

We had tried [1] to express the tensor commutation matrix $U_{2 \otimes 3}$ and $U_{3 \otimes 2}$ as linear combinations of the tensor products of the Pauli matrices with

the 3×3 -Gell-Mann matrices, in expecting to have expressions that lead to the generalization of (1.1). However, the obtained expressions are not interesting enough for the wanted generalization. We have noticed that for generalizing (1.1) to the expression of $U_{n \otimes p}$, $n \neq p$, we should use rectangle matrices instead of square matrices. We call rectangle Gell-Mann matrices such rectangle matrices.

At first, we talk about the tensor commutation matrices. After that, we construct the $n \times p$ -Gell-Mann matrices, in starting with some examples.

Let us denote by $I_{n \times p}$ the $n \times p$ matrices obtained in adding into the unit matrix $I_{\inf(n,p)}$ $|n - p|$ rows or $|n - p|$ columns formed by zeros.

2 Tensor commutation matrix

Definition 2.1. For $n, p \in \mathbb{N}$, $n \geq 2$, $p \geq 2$, we call tensor commutation matrix $n \otimes p$ the permutation matrix $U_{n \otimes p} \in \mathcal{M}_{np \times np}(\mathbb{C})$ formed by 0 and 1, verifying the relation

$$U_{n \otimes p} \cdot (a \otimes b) = b \otimes a$$

for all $a \in \mathcal{M}_{n \times 1}(\mathbb{C})$, $b \in \mathcal{M}_{p \times 1}(\mathbb{C})$.

We can construct $U_{n \otimes p}$ by using the following rule [5].

Rule 2.2. Let us start in putting 1, at the first row and first column, after that let us pass into the second column in going down at the rate of n rows and put 1 at this place, then pass into the third column in going down at the rate n rows and put 1, and so on until there is only $n - 1$ rows for going down (then we have obtained as numbers of 1: p). Then, pass into the next column which is the $(p+1)$ -th column, put 1 at the second row of this column (second row, because $(n - 1) + 1 = n$) and repeat the process until we have $n - 2$ rows for going down (then we have obtained as numbers of 1: $2p$). After that, pass into the next column which is the $(2p+1)$ -th column, put 1 at the third row of this column (third row, because $(n - 2) + 2 = n$) and repeat the process until we have $n - 3$ rows for going down (then we have obtained as number of 1: $3p$). Continuing in this way we will have that the element at $n \times p$ -th row and $n \times p$ -th column is 1. The other elements are 0.

Proposition 2.3. For $n, p \in \mathbb{N}$, $n, p \geq 2$,

$$U_{n \otimes p} = \sum_{(i,j)}^{(p,n)} E_{p \times n}^{(i,j)} \otimes E_{p \times n}^{(i,j)^t} = \sum_{(i,j)}^{(p,n)} E_{p \times n}^{(i,j)} \otimes E_{n \times p}^{(j,i)}$$

where $E_{p \times n}^{(i,j)}$ is the elementary $p \times n$ -matrix formed by zero except the element at i -th row, j -th column which is equal 1.

Proof. Let $a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathcal{M}_{n \times 1}(\mathbb{C})$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix} \in \mathcal{M}_{p \times 1}(\mathbb{C})$

$$\begin{aligned}
U_{n \otimes p} \cdot (a \otimes b) &= \sum_{(i,j)}^{(p,n)} E_{p \times n}^{(i,j)} \otimes E_{n \times p}^{(j,i)} \cdot (a \otimes b) \\
&= \sum_{(i,j)}^{(p,n)} (E_{p \times n}^{(i,j)} \cdot a) \otimes (E_{n \times p}^{(j,i)} \cdot b) \\
&= \sum_{(i,j)}^{(p,n)} (\delta_{ik} a_j)_{1 \leq k \leq p} \otimes (\delta_{jl} b_i)_{1 \leq l \leq n} \\
&= \sum_{(i,j)}^{(p,n)} (\delta_{ik} b_i)_{1 \leq k \leq p} \otimes (\delta_{jl} a_j)_{1 \leq l \leq n} \\
&= \sum_{(i,j)}^{(p,n)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&= b \otimes a
\end{aligned}$$

Example 2.4. The application of the rule yields us

$$\begin{aligned}
U_{2 \otimes 3} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
U_{2 \otimes 3} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

□

3 Construction of a system of rectangle Gell-Mann matrices

At first, let us consider some particular cases.

3.1 $U_{2 \otimes 3}$

The 2×2 -Gell-Mann matrices are the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Being inspired by a way for constructing the $n \times n$ -Gell-Mann matrices from the $(n-1) \times (n-1)$ -Gell-Mann matrices, where on the diagonal there is not zero between two non zero elements and the first non zero element is the first element (that is, the element at the first row, first column) (Cf. for example[6]), we add into these four matrices third column formed by zeros. Then, we have a system formed by

$$I_{2 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix}, \Lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

And for obtaining a basis of $\mathcal{M}_{2 \times 3}(\mathbb{C})$, we introduce the matrices

$$\Lambda_4 = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \Lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \text{ We can check easily that}$$

$$U_{2 \otimes 3} = \frac{1}{2} I_{2 \times 3}^+ \otimes I_{2 \times 3} + \frac{1}{2} \sum_{a=1}^5 \Lambda_a^+ \otimes \Lambda_a$$

where Λ_a^+ is the hermitian conjugate of Λ_a .

3.2 $U_{3 \otimes 2}$

Using analogous way, but this time we are adding into the Pauli matrices and σ_0 a third row formed by zeros, instead of column. Then, we get a system $(\Lambda_a)_{1 \leq a \leq 5}$ of 3×2 matrices which satisfies

$$U_{3 \otimes 2} = \frac{1}{2} I_{3 \times 2}^+ \otimes I_{3 \times 2} + \frac{1}{2} \sum_{a=1}^5 \Lambda_a^+ \otimes \Lambda_a$$

In fact, $U_{n \otimes p} = U_{n \otimes p}^t = U_{n \otimes p}^+$, for all $n, p \in \mathbb{N}$, $n, p \geq 2$.

3.3 $U_{2\otimes 4}$

Using yet the analogous way, but in this case we are adding into the Pauli matrices and the 2×2 unit matrix third and fourth columns formed by zeros. Then, we have a system formed by four 2×4 matrices $I_{2 \times 4}, \Lambda_1, \Lambda_2, \Lambda_3$. And for obtaining a basis of $\mathcal{M}_{2 \times 4}(\mathbb{C})$, we introduce the matrices

$$\Lambda_4 = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Lambda_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}, \Lambda_6 = \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \text{ in the system. The system satisfies the relation}$$

$$U_{2\otimes 4} = \frac{1}{2} I_{2 \times 4}^+ \otimes I_{2 \times 4} + \frac{1}{2} \sum_{a=1}^7 \Lambda_a^+ \otimes \Lambda_a$$

3.4 $U_{3\otimes 4}$

In this case, we start from the 3×3 -Gell-Mann matrices. Then, we have a system $(\Lambda_a)_{1 \leq a \leq 11}$ of 3×4 matrices which satisfies the relation

$$U_{3\otimes 4} = \frac{1}{3} I_{3 \times 4}^+ \otimes I_{3 \times 4} + \frac{1}{2} \sum_{a=1}^{11} \Lambda_a^+ \otimes \Lambda_a$$

Definition 3.1. Let $n, p \in \mathbb{N}$, $p \geq n \geq 2$. We call $n \times p$ -Gell-Mann matrices the $n \times p$ matrices $\Lambda_1, \Lambda_2, \dots, \Lambda_{n^2-1}, \Lambda_{n^2}, \Lambda_{n^2+1}, \dots, \Lambda_{np-1}$ such that:

$\Lambda_1, \Lambda_2, \dots, \Lambda_{n^2-1}$ are obtained in adding into the $n \times n$ -Gell-Mann matrices $(n+1)$ -th, $(n+2)$ -th, \dots , p -th columns, formed by zeros;

$$\Lambda_{n^2} = \sqrt{2} E_{n \times p}^{(1, n+1)}, \Lambda_{n^2+1} = \sqrt{2} E_{n \times p}^{(2, n+1)}, \dots, \Lambda_{n^2+n-1} = \sqrt{2} E_{n \times p}^{(n, n+1)},$$

$$\Lambda_{n^2+n} = \sqrt{2} E_{n \times p}^{(1, n+2)}, \Lambda_{n^2+n+1} = \sqrt{2} E_{n \times p}^{(2, n+2)}, \dots, \Lambda_{n^2+2n-1} = \sqrt{2} E_{n \times p}^{(1, n+2)},$$

$$\dots, \Lambda_{n(p-1)} = \sqrt{2} E_{n \times p}^{(1, p)}, \Lambda_{n(p-1)+1} = \sqrt{2} E_{n \times p}^{(2, p)}, \dots, \Lambda_{np-1} = \sqrt{2} E_{n \times p}^{(n, p)}.$$

Then, we define the $p \times n$ -Gell-Mann matrices as the matrices obtained in taking the hermitian conjugates of the $n \times p$ -Gell-Mann matrices.

Proposition 3.2. For $n, p \in \mathbb{N}$, $p, n \geq 2$, consider the system of $n \times p$ -Gell-Mann matrices $\Lambda_1, \Lambda_2, \dots, \Lambda_{np-1}$. Then,

$$U_{n \otimes p} = \frac{1}{\inf(n, p)} I_{n \times p}^+ \otimes I_{n \times p} + \frac{1}{2} \sum_{a=1}^{np-1} \Lambda_a^+ \otimes \Lambda_a \quad (3.1)$$

Proof. Let us suppose $p \geq n$.

$$\frac{1}{2} \sum_{a=n^2}^{np-1} \Lambda_a^+ \otimes \Lambda_a = \sum_{(j, l)=(1, n+1)}^{(n, p)} E_{n \times p}^{(j, l)^t} \otimes E_{n \times p}^{(j, l)} \quad (3.2)$$

Using the proposition and the formula (1.1) we have

$$\sum_{(j,l)=(1,1)}^{(n,n)} E_{n \times n}^{(j,l)t} \otimes E_{n \times n}^{(j,l)} = \frac{1}{n} I_n \otimes I_n + \frac{1}{2} \sum \Lambda_a^{(n)} \otimes \Lambda_a^{(n)} \quad (3.3)$$

In adding, in (3.3), into the terms on the left of \otimes 's $p - n$ rows, $(n + 1)$ -th, $(n + 2)$ -th, ..., p -th rows, and on the right $p - n$ columns, $(n + 1)$ -th, $(n + 2)$ -th, ..., p -th columns, formed by zeros, the non zero elements which have same position, same row and same column, will keep same position. So by the definition of tensor product of matrices

$$\sum_{(j,l)=(1,1)}^{(n,p)} E_{n \times p}^{(j,l)t} \otimes E_{n \times p}^{(j,l)} - \sum_{(j,l)=(1,n+1)}^{(n,p)} E_{n \times p}^{(j,l)t} \otimes E_{n \times p}^{(j,l)} = \frac{1}{n} \Lambda_0^+ \otimes \Lambda_0 + \frac{1}{2} \sum_{a=1}^{n^2-1} \Lambda_a^+ \otimes \Lambda_a$$

Using the proposition and (3.2) we have (3.1). \square

Now, we are giving some properties of the rectangle Gell-Mann matrices.

Proposition 3.3. *For $n, p \in \mathbb{N}$, $p, n \geq 2$, let $(\Lambda_a)_{1 \leq a \leq np-1}$ a system of $n \times p$ -Gell-Mann matrices. Then, $Tr(\Lambda_a^+ \Lambda_b) = 2\delta_{ab}$ where δ_{ab} is the Kronecker symbol.*

Proposition 3.4. *For $n, p \in \mathbb{N}$, $p \geq n \geq 2$, let $(\Lambda_a)_{1 \leq a \leq np-1}$ a system of $n \times p$ -Gell-Mann matrices. Then,*

$$\Lambda_a \Lambda_b^+ - \Lambda_b \Lambda_a^+ = i \sum_{c=1}^{n^2-1} f_{abc} \Lambda_c^{(n)}$$

where f_{abc} 's the components of a tensor totally antisymmetric, with $f_{abc} = 0$ if at least one of a, b, c is in $\{n^2, n^2 + 1, \dots, np - 1\}$.

Conclusion

Being inspired by a way for constructing the $n \times n$ -Gell-Mann matrices from the $(n - 1) \times (n - 1)$ -Gell-Mann matrices, where on the diagonal there is not zero between two non zero elements and the first non zero element is the first element (that is, the element at the first row, first column), we can construct a basis of $\mathcal{M}_{n \times p}(\mathbb{C})$, whose elements make generalization of the expression of $U_{n \otimes n}$ in terms of the tensor products of $n \times n$ -Gell-Mann matrices to the expression of $U_{n \otimes p}$.

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